CONVEXITY OF A SHOCK WAVE IN THE SUBSONIC SEGMENT IN PLANE FLOW

PMM Vol. 33, No. 1, 1969, pp. 158-162
E.G. SHIFRIN
(Moscow)
(Received July 10, 1968)

We consider the topological properties of flow in the subsonic region situated behind the shock wavefront. We prove that when a supersonic jet enters a region of increased pressure with irregular reflection of the oblique condensation jump from the symmetry axis, then the Mach shock wavefront has its concave side towards the oncoming stream. Conversely, if a shock wave in a supersonic flow forms in front of a blunt body immersed in this flow, then the segment responsible for retarding the gas to subsonic velocities is convex. Both these properties follow from the results of various experiments and computations, although so far no strict proof of the above statements has been offered.

Let us consider a plane flow behind a smooth shock wave in a uniform supersonic stream.

The equations of plane adiabatic flow can be written in a local coordinate system related to the streamlines,

$$(1 - M^2)\frac{\partial p}{\partial s_1} = kpM^2\frac{\partial \beta}{\partial s_2}, \qquad \frac{\partial p}{\partial s_2} = -kpM^2\frac{\partial \beta}{\partial s_1} \tag{1}$$

where p is pressure; β is the angle of inclination of the velocity vector measured anticlockwise; M is the Mach number; k is the ratio of specific heats (we assume that $k > 1$), $\partial(\ldots)/\partial s_1$, $\partial(\ldots)/\partial s_2$ are the directional derivatives along and normal to the velocity vector; the latter direction s obtained by rotating the velocity vector by $\frac{1}{2}\pi$ anticlockwise.

Let us consider the mapping of the region situated behind the shock wave in the physical xy -plane, into the $p\beta$ -plane (the β - and y -axes are directed vertically upwards and the p - and x-axes - horizontally to the right). Using (1), we obtain the following expression for the Jacobian of $\partial(p, \beta)/\partial(x, y)$

$$J = \frac{\partial(p, \beta)}{\partial(x, y)} = \frac{\partial(p, \beta)}{\partial(s_1, s_2)} = \frac{1}{kpM^2}\left[(1 - M^2)\left(\frac{\partial p}{\partial s_1}\right)^2 + \left(\frac{\partial p}{\partial s_2}\right)^2\right] \tag{2}$$

Since $J \geqslant 0$ for $M \leqslant 1$, the mapping of the subsonic region into the $p\beta$-plane has no folds, because passage through the edge of a fold (branch line) would alter the sign of the Jacobian. (Nikol'skii and Sedov obtained this property in a somewhat different form in [1]). Further, the point at which $J = 0$ is isolated when $M < 1$, since the solution of the Cauchy problem with the initial conditions given at the line $J = 0$ would lead to the trivial case of a uniform flow. (It should be noted, however, that a mapping which is locally univalent need not be so "in the whole". Projection onto a spiral surface with its axis deleted is an example of this).

Since the Jacobian J represents the ratio of areas of the oriented elementary contours in the planes $p\beta$ and xy, its nonnegativity for $M \leqslant 1$ implies the following rule.

When the boundary G of the subsonic region is traversed anticlockwise, with the region remaining to the left, the boundary of its image in the $p\beta$-plane is also traversed anticlockwise, with the image also remaining to the left.

The image of the shock wave in the $p\beta$-plane lies on the closed curve $\beta = \beta(p)$ where the function $\beta(p)$ is given by the relations at the condensation jump. Let us turn our attention to the "subsonic" segment bsb' (Fig. 1) lying on this curve (i.e. on the shock polar); as we move along this shock polar from the point s to the point n, the velocity increases and the pressure decreases (s corresponds to the point S of orthogonality of the shock wave to the velocity vector; n corresponds to the point at which the shock wave degenerates into a characteristic).

Property 1. Let us denote by T the point of inflection of the shock wave relative to the exterior normal. If a point T exists on the shock wave, then the relations at the condensation jump imply that traversal of the shock wave past the point T, corresponds to traversal of the shock polar with the cusp t, which is the image of T (Fig. 2). If the velocity at T is subsonic, then the fact that $J \geqslant 0$ when $M \leqslant 1$ implies that the image of the shock wave near T is a cut in the image of the region behind the shock wave (Fig. 2).

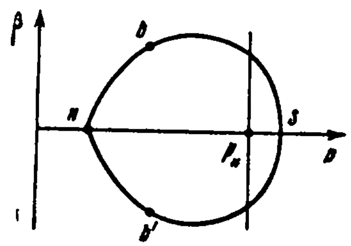

Fig. 1

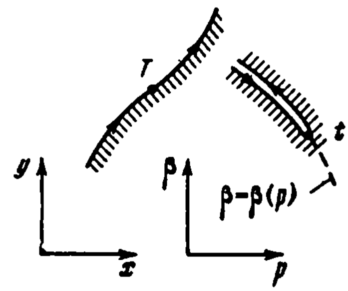

Fig. 2

Property 2. Analysing the formula for the pressure behind the condensation jump, we find that when $1 < M_\infty < \nu(k)$ then the shock polar intersects the straight line

$$p = p_* = p_\infty \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

The points of intersection belong to the subsonic segment of the shock polar. (Here M_∞ and p_∞ are the Mach number and the total pressure in the incident flow; $\nu(k)$ is a constant; in addition we shall denote the Mach number at the points of intersection of the shock wave and the straight line $p=p_*$ by M_* and the minimum value of the Mach number on the shock polar by M_s). This implies that when $1 < M_\infty < \mu(k, M_*)$, where $\mu(k, M_*)$ is constant and smaller than $\nu(k)$ (Fig. 3), the segment of the shock polar on which $M_s \leqslant M \leqslant M_*$ is situated to the right of the straight line $p = p_*$, i.e. that on this segment

$$p \geqslant p_\infty \left(\frac{2}{k+1}\right)^{k/(k-1)} = p_*$$

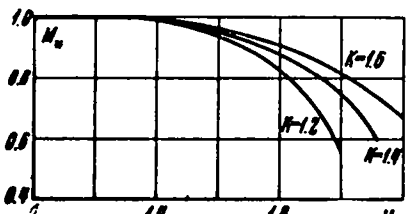

Fig. 3

Property 3. Since on the sonic line we have

$$p = p_0 \left(\frac{2}{k+1}\right)^{k/(k-1)} < p_\infty \left(\frac{2}{k+1}\right)^{k/(k-1)} = p_*$$

(p_{01} denotes the total pressure in the region behind the shock wave), the image of the sonic line in the $p\beta$ plane is situated to the left of the straight line $p = p_*$.

Let us denote the subsonic region adjacent to the shock wave by G. In general, this region can be bounded by segments of the shock wave, of the tangential discontinuities (which are absent when the shock wave is smooth), by the segments of the profile contours, by the sonic lines, and by the secondary condensation jumps in the locally supersonic zones.

Let us limit ourselves to cases where the boundary of G does not contain secondary condensation jumps, and consider two types of G : those whose boundary contains a segment of the profile contour, and those whose boundary does not contain such a segment..

Theorem 1. Let the boundary of the subsonic region G consist only of the segments of the smooth shock wave and of sonic lines. For $M_\infty < \mu(k, M_*)$ the segment of the shock wave (belonging to the boundary of G) on which $M < M_*$ is convex towards the region behind the shock wave at each of its points (Fig. 4).

Let us assume the opposite and postulate the existence of a point T on the above segment of the shock wave. Then a subregion F of G exists whose image in the $p\beta$ - plane is situated outside the loop of the shock polar and to the right of the line $p = p_t$. This follows from the Property 1 if the image of the shock wave

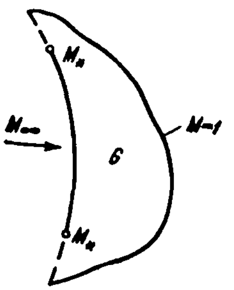

Fig. 4

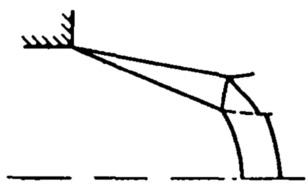

Fig. 5

near T lies on the shock polar to the left of t. If, on the other hand, the image of the shock wave appears to the right of t, on the shock polar, then the image of F is near s, which is the extreme right-hand point of the shock polar. Thus we can state without loss of generality that no other point t exists between the points t and s on the shock polar. Since the pressure in the flow is finite, the image of F must be bounded on the right.

Since the boundary of G consists only of segments of the shock wave and of sonic lines, and since there are no branch lines for $M \leq 1$ (if the mapping includes folds the boundary of the image may contain a branch line), the boundary of the image of at whose points $p > p_t$ can only be the mapping of a segment of the sonic line. By Properties 2 and 3 this is impossible when $M_\infty < \mu(k, M_*)$.

Thus we find that the shock wave does not contain any points T on a certain subsonic segment (for $M < M_*$) and that the image of a neighborhood of this segment lies inside the loop of the shock polar. Using the rule of passage around the boundary of G and the relations at the condensation jump, we find that the shock wave on this segment is convex towards the region behind the shock wave.

Such a flow with a "concave" shock wave occurs when a supersonic jet flows into a region of high pressure and the oblique condensation jump is reflected "irregularly" from the axis of symmetry of the jet [9] (Fig. 5).

Now let us consider a region G whose boundary includes a segment of the profile contour. Let us investigate a flow past a smooth convex profile with a receded shock wave. The critical point is that at which the streamline reaching the profile branches out into two streamlines; the velocity at the critical point is zero.

Theorem 2. Let the following properties hold for a supersonic flow past a smooth convex profile which is the only object in the way of the stream:

1. The shock wave is smooth over the whole of its length.
2. The boundary of the subsonic region G does not contain secondary condensation jumps.
3. The critical point at the profile is unique.

If $M_\infty < \mu(k, M_*)$, then the segment of the shock wave (lying on the boundary of G) on which $M < M_*$ is convex towards the oncoming flow at each of its points.

Let us turn our attention to the mapping into the $p\beta$-plane. Since the profile is smooth, the image of the critical point O is the segment O_1O_2 of length π of the straight line $p = \text{const}$ (Fig. 6). The profile contour is mapped into curves continuing this segment at both ends. The convexity of the profile means that the angle β increases along the curve beginning at O_1 and decreases along the curve beginning at O_2 (Fig. 6).

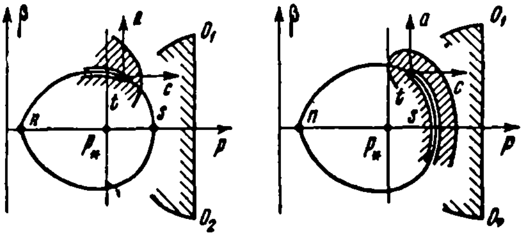

Fig. 6

The segment O_1O_2 lies outside the loop of the shock polar and to the right of it (Fig. 6). At the point O we have

$$p = p_0 \geqslant p_{0s}$$

Equality applies only when the streamline which passes through the point S at which the shock wave is orthogonal to the velocity vector also passes through the critical point. At the point s we have

$$p = p_{0s}\left(1 - \frac{k-1}{k+1}\lambda_s^2\right)^{k/(k-1)} < p_{0s}$$

so that the pressure at s is smaller than that at O. Here p_{0s} and λ_s denote the total pressure and velocity coefficient at the point S, and $\lambda_s > 0$.

The image of the shock wave is the entire loop of the shock polar. The points T are either absent from the shock polar, or their number is even. Indeed, since the shock wave degenerates near infinitely distant points into the characteristics of various families (into the characteristic of the first (second) family in the upper (lower) half-plane), the images of these neighborhoods in the plane $p\beta$ are segments of the shock polar, bounded by the point n below and above, respectively. Since the shock wave is smooth, its mapping is a continuous curve. If points T exist on the shock wave, then points t (images of T) break up the shock polar into segments which are traversed an odd number of times, different for each segment, in a single traversal of the shock wave.

Let us now assume that a point T lies on the segment of the shock wave referred to in Theorem 2. We can say without loss of generality that the image t of this point lies in the upper half-plane of $p\beta$, i.e. that $\beta_t \geqslant 0$. The point t bounds a segment which is traversed (with traversal of the shock wave) not less than three times. Using the rule of traversing the boundary of the subsonic region, we find that not less than two sheets of the image of the subsonic region G exist near the point t outside the loop of the shock polar. These sheets are shown in Fig. 6 in two possible variants, depending on whether the image of the shock wave near T lies to the left or to the right of t. Since branch lines are absent when $M \leqslant 1$, the sheets are not joined to each other.

Since both sheets contain the upper right-hand quadrant atc of the neighborhood of t (situated outside the loop of the shock polar), the boundaries of these sheets intersect this quadrant. Hence, a segment of the boundary of the subsonic region G, which does not belong to the shock polar, exists on each sheet to the right of t (i.e. $p > p_t$ on this segment). When this segment is traversed in the direction of increasing β, the image of G (in some neighborhood of this segment) lies to the left of it. Using again the rule of traversing the boundary of the subsonic region, we find that the image of the profile contour does not satisfy this condition because it is convex, while the image of the sonic line fails to satisfy it because of properties 2 and 3. This leaves only the image O_1O_2 of the point O. However, this segment can bound only one sheet of the image G, since we have no branch lines when $M \leqslant 1$.

This contradiction proves that our assumption concerning the existence of the point T on the indicated segment of the shock wave was invalid. The image of G near this segment lies outside the loop of the shock polar. Applying the rule traversing the boundary of G, we find that the shock wave is convex in the direction of the oncoming flow on this segment.

Let us now consider flow past a sharp convex profile with an attached shock wave, when the flow behind it is subsonic in some neighborhood of the tip. Analyzing the shock polar, we find that such a state may exist when the ratio between M_∞ and the angle of inclination of the profile (at the tip) to the velocity vector of the oncoming flow assumes a certain value.

Theorem 3. Let the following conditions be fulfilled in a supersonic flow past a sharp convex profile with an attached shock wave:

1. The shock wave is smooth everywhere beginning the point of attachment.
2. The boundary of the subsonic region G near the apex does not contain any secondary condensation jumps.

If $M_\infty < \mu(k, M_*)$, then the segment of the shock wave on which $M < M_*$ (provided that boundary of G contains such a segment) is convex towards the oncoming flow at all its points.

The proof of this theorem is analogous to that of Theorem 2.

BIBLIOGRAPHY

1. Sedove, L.I., Plane Problems of Hydro- and Aerodynamics. M., Gostekhizdat, 1950.
2. Guderley, K.G., Theory of Transonic Flows. M., Izd. inostr. lit., 1960.

Translated by L.K.